PD CF 0304

National
Aerospace
Laboratories

A Note on the No-Slip Boundary Condition

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Project Document CF 0304

June 2003
Bangalore 560017, India

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1. The no-slip boundary condition

It is known now, beyond any doubt, that a moving fluid in contact with a solid body will not have any velocity relative to the body at the contact point. This condition of not slipping over a solid surface has to be satisfied by a moving fluid. This is known as the no-slip condition and is stated routinely in the text books of Fluid Mechanics. But it remained a difficult problem for a long time. We will first give in this article some basic ideas connected to this problem so that the historical notes added afterwards will be appreciated better. Some recent experimental data of interest, the phenomenon at the molecular level and the case of turbulent flows will be discussed briefly. We will see that this simple looking phenomenon was so difficult to comprehend and even the giants had to struggle. The students today are taught it in one stroke. It is not surprising that some of them get bothered about it. If one did not, we will see why one should be.

2. What Happens to a Fluid Particle at the Wall?

Before considering the case of fluids, i.e. gases and liquids, consider a simpler case of an isolated ball. When the ball hits a wall of a solid body, its velocity abruptly changes. This abrupt change in momentum of the ball is achieved by an equal and opposite change in that of the wall or the body. Thus the overall momentum is conserved in each of the three directions.

Now assume that the ball is spherical and nearly rigid and it impinges on a smooth rigid wall at an angle. See Figure 1. If the wall is heavy its motion can be neglected. At the point of collision we can identify normal and tangential directions $\hat{n}$ and $\hat{t}$ to the wall. The time of impact t_o is very brief. It is a good assumption to conclude that the normal velocity V_n will be reversed with a loss in magnitude since these two bodies cannot be exactly rigid. Further, if we assume the time of impact to be zero, the normal velocity component V_n is seen to be discontinuous and also with a change in sign. See figure 2. Whether it is discontinuous or not, the fact that it has to change sign is obvious, since the ball cannot continue penetrating inside the solid wall. The case of the tangential component V_τ is far more complex and hence more interesting. First of all, the ball will continue to move in the same direction and hence there is no change of sign. If the wall and the ball are perfectly smooth (i.e. frictionless) V_τ will not change at all. But in case of rough surfaces V_τ will decrease a little. But it is important to note that V_τ is no-where zero. This is true even when we relax the assumptions made in this model. Even though the ball sticks to the wall for a brief period t_o, at no time is its velocity zero!

The ball can also roll on the surface. If we assume the ball to be spherical and rolling on the surface without slip the contact point is the instantaneous centre of rotation and it is not moving at that instant. The contact point keeps changing as the ball rolls. Friction is required to avoid slip but it does not assure that slip is completely prevented. There is no constraint from any considerations for the slip to be zero. One can imagine that if the velocity increases there is likely to be some slip. We will see the implication of this for fluid motion later. Also for a body of irregular shape or on an irregular surface smooth rolling motion without slip is not possible.

The problem of fluids is considered now. This is fundamentally different from the case of an isolated ball since a *flow field* has to be considered now. The difference is that a fluid element in contact with a wall also interacts with the neighboring fluid. Once we recognize this difference, the problem appears too difficult and the previous model of an isolated ball impinging or rolling on the wall is not of much help. It is not surprising. It was only at the end of the 19th century that this problem was resolved using both theoretical and experimental tools. Even though the problem was considered prior to the 19th century, during the whole of that century extensive work was required to resolve the issue.

Even though we agreed that the case of a simple ball is not adequate here, an idea used there can still be applied here. The idea is that the normal component of velocity at the solid wall should be zero to satisfy no penetration. Quite interestingly in case of fluids the tangential velocity is also zero at the wall. This is the so-called no-slip boundary condition and we will see how different it is compared to the simple ball impingement case. Before giving the details and a historical perspective some details of the fluid motion are given in the next section.

3. Continuum Hypothesis and the Navier-Stokes Equations

Since the number of molecules in a fluid is very large, it is possible to ignore the existence of the individual molecules and consider the fluid to be homogeneous and of uniform properties. This is much simpler than considering the dynamics of the molecules. This is the so-called continuum hypothesis. In this model the fluid does not have any voids like intermolecular spaces. The classical laws of motion, of course, apply to this fluid. We talk of the fluid elements or fluid particles which deform in the flow. Forces acting on these elements determine the acceleration. But the forces consist of both the externally applied forces like gravity or magnetic field and also the internal stresses. The stresses acting on a fluid element are determined by the *rate* of deformation of the element. This is where one faces the difficulty. How to relate stresses to the rate of deformation or velocity components?

This was done independently in the first half of the 19th century by the French engineer Navier (1785 – 1836) and the Irish mathematician and physicist Stokes (1819 – 1903). They derived the well known equations of motion known now as the Navier-Stokes equations and which relate the acceleration of the fluid element to the net force acting in each direction. We need the appropriate boundary conditions to solve these equations. Quite interestingly these equations helped in resolving the uncertainty about the no-slip boundary condition. These equations will be given in the next section.

Two other ideas are relevant here. The first is about the nature of fluid stress. When the fluid is at rest, only the normal stresses are exerted, tangential stresses being zero. The normal stress at a point does not depend on the direction and it is the hydro-static pressure. When the fluid is in motion, the pressure changes from this hydro-static value and also additional tangential stresses are induced. It suffices for the present purpose to know that these additional stresses are obtained by multiplying the rate of deformation (which is related to spatial velocity derivatives) by the viscosity of the fluid. For common fluids like water and air this linear relation between the stress and rate of strain or deformation is a good approximation and such fluids are known to be newtonian.

The second idea is about the specification of the velocity field. Two alternatives are possible. In the first, or the so-called Lagrangean description, we extend the idea from particle mechanics. Here the velocity is associated with distinct pieces of matter that are identified like a particle or ball. In the second description, known as the Eulerian description, the velocity is associated with a location in the flow but not any distinct matter. Hence when we say x-component of velocity u at location (x, y, z) and time t i.e. $u(x, y, z, t)$ it pertains to a location and hence to different pieces of fluid occupying this location at the instant considered. Both these descriptions of velocity are used in the study of fluid motion depending on the context, but the Eulerian description is more common. This directly gives the spatial gradients needed to calculate the stress. A fixed probe meant to measure the velocity like a pitot tube or a hot-wire probe or a laser Doppler anemometer measures the Eulerian velocity. We will use this description only.

4. Some Details and Simplifications.

Even though we described the N-S equations in the previous section, their mathematical form was not given. It is possible to read this article without considering this exact form. However, considering the details will be more fruitful. Since the N-S equations are solved along with the mass conservation or the continuity equation we give that equation also (equation (4.1) below). To simplify matters we consider flow only in two directions (x, y). Let t be time, (u,v) velocity components along (x, y) and p be the pressure. Further the fluid is assumed to be incompressible with density r and viscosity m. Then

$$u_x + v_y = 0 \tag{4.1}$$

$$r(u_t + uu_x + vu_y) = -p_x + m(u_{xx} + u_{yy}) \tag{4.2}$$

$$r(v_t + uv_x + vv_y) = -p_y + m(v_{xx} + v_{yy}) \,. \tag{4.3}$$

Here the subscripts indicate partial differentiation. Hence $u_t = \partial u / \partial t,\ u_x = \partial u / \partial x,$
$v_{xx} = \partial^2 v / \partial x^2$ etc. In equations (4.2) and (4.3) the left hand side represents the acceleration of a fluid element and the right hand side the net force on it.

This system of three equations has three dependent variables (u, v, p) and can be solved if appropriate boundary conditions are specified. Specification of the boundary conditions is a mathematically difficult issue and we will not deal with that in detail. Also, notice that this system is second order in space because of terms like u_{xx} on the right hand side and is non-linear because of the nonlinear (or second power in dependent variables) terms like uu_x on the left hand side.

If we neglect the viscosity of the fluid the second term in equations (4.2) & (4.3) will be dropped. Then what are left are the Euler equations of motion. These inviscid equations are first order in space and are still non-linear. These were derived by the Swiss mathematician Euler (1707-1783) before Navier and Stokes gave the equations for a real (that is viscous) fluid.

An interesting observation about this simplification: When we dropped the viscous terms to get the Euler equations, the order of the equations decreased by one. This should also translate into a reduced requirement on the boundary conditions. And that is exactly what happens. For Euler equations we need only the specification of the normal component of the velocity (to be zero). The solution of the Euler equation leads to a slip velocity at the wall. For the N-S equations, which are one order higher, we have to specify the tangential component also. Note that there is no need for it to be zero but its precise description is required. For an interesting discussion of this point see Arakeri & Shankar (2000).

5. The Hagen-Poiseuille Flow

This is the fully developed laminar flow in a long tube of circular cross-section. We discuss it here since it will be referred to frequently in the rest of the article and also because it was very helpful in the experimental verification of no-slip. The need for specification of the tangential velocity on the wall will be specially highlighted. For mathematical simplicity we consider a 2-D planar flow rather than through an axisymmetric tube. The mathematical details and the qualitative results are similar in both the cases. Final results will be given for the axisymmetric tube case also. Mathematically minded readers will be benefited by some of the details given below. Those who are not interested in the details can skip a paragraph and go straight to the final results in this section.

In a steady flow all derivatives with respect to time t and in a very long tube all derivatives with respect to x will go to zero (see figure 3). Hence from equation (4.1) $v_y = 0$ leading to v = constant. This constant is zero since v, the normal velocity on the wall at $y = h$ is zero. Now each of the terms on the LHS of equations (4.2) and (4.3) is zero. And equation (4.3) reduces to $p_y = 0$ leading to $p = p(x)$ i.e. in the entire cross-section p is constant. Equation (4.2) simplifies to $p_x = mu_{yy}$. Notice that p_x is the total derivative dp/dx and further since u_{yy} cannot depend on x, p_x should be independent of x, hence a constant. This equation can be integrated twice. It is here that we have to specify the tangential velocity on the wall. Whether there is slip or no-slip is unimportant in the solution of the equation but its precise specification is mandatory. This flow gives us an excellent opportunity to measure the slip if there is any.

Integrating $p_x = mu_{yy}$ w.r.t. y we get

$$u_y = \frac{p_x}{m} y + A \tag{5.1}$$

$$u = \frac{p_x}{2m} y^2 + Ay + B \,. \tag{5.2}$$

Because of symmetry in y, $A = 0$. The other constant B is fixed by the value of u on the wall at $y = h$. If Δu_w is the assumed slip at the wall,

$$B = \Delta u_w - \frac{p_x}{2m} h^2 \tag{5.3}$$

$$u = -\frac{1}{2m} \frac{dp}{dx} (h^2 - y^2) + \Delta u_w . \tag{5.4}$$

Since pressure is decreasing along x, p_x is negative.

The corresponding equations for a tube (see fig 3) with radius R are –

$$u = -\frac{1}{4m}\frac{dp}{dx}\left[R^2 - r^2\right] + \Delta u_w \tag{5.5}$$

$$u_r = \frac{r}{2m}\frac{dp}{dx} \tag{5.6}$$

$$Q = 2p\int_o^R urdr = p\left[\frac{R^4}{8m}\left(-p_x\right) + R^2\Delta u_w\right] \tag{5.7}$$

where Q is the flow rate. Note that the presence of velocity slip at the wall does not change the shear stress distribution and also the discharge Q increases due to slip for a given pressure drop.

It is very tempting to conclude by looking at these equations that we can measure the slip or at least decide whether the slip is there at all. But it is not so simple. If we define the resistance coefficient l for the tube by

$$l = \left(-p_x\right)\frac{2R}{\frac{1}{2}r\bar{u}^2} \tag{5.8}$$

and Reynolds number $\mathrm{Re} = \frac{2Rr\bar{u}}{m}$,
where the average velocity $\bar{u} = Q/\left(pR^2\right)$ we get

$$l = \frac{64}{\mathrm{Re}}\left(1 - \frac{\Delta u_w}{\bar{u}}\right). \tag{5.9}$$

A graph of l vs. Re on a log-log plot with $\Delta u_w = 0$ is a straight line with slope $= -1$ (Figure 4). A constant value of $\Delta u_w / \bar{u}$ will shift the graph but the slope will remain constant. Hence it will be difficult to identify if the slip is present.

It may be added that Poiseuille was led to this study to understand the blood flow and recommended that the hydraulic engineers should study the motion of particles in moving liquids with the aid of a microscope (1846). Hagen assumed zero velocity at the wall in an earlier paper but later (1839) adopted the idea of a stagnant layer near the wall but without slip.

6. Dry Friction

It is appropriate to recapitulate here our knowledge about the dry friction that occurs when a solid surface slides over another dry solid surface. Conceptually the ideas in this case may appear simpler compared to the fluid case but getting reliable quantitative data is very difficult. The surface conditions affecting dry friction may not be uniform and may even depend on the direction of motion. On the other hand, fluid properties for simple fluids like air and water are more uniform and experiments with the fluid friction become more repeatable. But the physics involved is more complex.

In the light of the comments made above, it is not surprising that the laws of dry friction were not developed too early in the human history even though it is likely that many had an intuitive feeling for them. It is the experiments of Coulomb in 1781 and those of Morin from 1831-1834 that played a decisive role in formulating the laws of dry friction or Coulomb friction. This period roughly coincides and slightly precedes the time when the laws of fluid friction were also being developed.

Imagine a solid block kept on a table and pushed gently sideward. When a dry surface has a tendency to slide over another similar surface held fixed, the normal forces on these two bodies at the contact surface balance each other. Also, the tangential force or the friction opposes the motion or the tendency to move. One can imagine that at the micro level the two surfaces are bound to have irregular ridges and valleys and contact each other only at select locations. The tendency of motion is opposed by these micro-irregularities and the opposing force at this level is not necessarily along the mean contact surface. The net frictional force is equal to the applied force as long as it is less than a limiting force and hence the body does not move. This limiting value is the maximum of static friction at impending motion. If the applied tangential force is greater than this threshold value, the body will start moving but then the frictional force known as the kinetic friction, is slightly less than the maximum static friction. Note that the contact locations at the micro level are continuously changing. One can imagine that when in motion it is the top parts of these micro-ridges that are in contact and this leads to a smaller tangential force. This leads to kinetic friction being smaller than the maximum static friction.

The maximum static friction is independent of the area of contact and is proportional to the normal contact force N between the two surfaces –

$$F_s(\max) = m_s N \tag{6.1}$$

where m_s is the coefficient of static friction. Similarly for the case of sliding we define the co-efficient of kinetic friction m_k by

$$F_k = m_k N \,. \tag{6.2}$$

Generally m_k is less than m_s as mentioned above.

Polishing a surface generally leads to a decrease in the dry friction. We will see some interesting contrast between this and the fluid friction.

7. Newton's Slip

One cannot imagine that a curious person like Isaac Newton would not have bothered about the motion of the fluids. He considered some discrete cases of fluid motion. In the three books in the Principia Newton (1725) dealt with vortex motion briefly in Book 2. His motivation was to see if the motion of a fluid vortex was consistent with the Keplerian planetary motion. Hence he considered only the circular motion. He was handicapped by not having the governing equations to describe the motion of either idealized or real fluids. But he did recognize that fluid resistance arose due to the velocity difference between two spatially separated points. The velocity difference is equivalent to velocity derivative in simple cases. Now we know that it is the rate of strain or rate of deformation that causes stress and the fluid resistance.

To study the vortex motion he considered an infinitely long circular cylinder immersed in an unbounded fluid and rotating about its axis at a uniform speed. The fluid is set into motion by the moving cylinder and the resulting streamlines are circular. Newton dealt with this problem from the first principles (i.e. not starting with any ready made equations) with the tacit assumption that there was no fluid slip at the cylinder wall. Unfortunately he obtained an incorrect expression for the velocity distribution. Still his conclusion that the motion of this vortex due to a rotating cylinder (also due to a sphere that he studied in the subsequent proposition) is not consistent with the Keplerian planetary motion turned out to be correct i.e. the velocity distribution along the radius in the vortex and also that of the planets in the solar system were not the same.

As mentioned above Newton correctly assumed that a rotating cylinder imparts the velocity to the fluid that is in contact without any slip. However, he missed a similar assumption in case of a projectile modelled by a cylinder moving forward in the direction of its length. He concluded that the resistance to motion depends on the diameter but not on the length of the cylinder. This erroneous conclusion that resistance is independent of length has the assumption that there is complete slip, i.e. the curved surface of the cylinder moves without affecting the fluid motion whatsoever. We should keep in mind that even if Newton had assumed that there was no-slip or only partial slip at the cylindrical surface it would not have been easy for him to get a relation for the drag dependence on the length of the cylinders. But it is very likely he would have then guessed correctly that the drag would increase with the length.

This historical note is added to emphasize how difficult it was to understand the motion of a fluid in contact with a solid body.

8. Historical Development

A brief and excellent review of this problem of velocity slip is given in the book by Goldstein (1957). We freely borrow from this book adding some explanations and supplements based on the earlier discussion.

We saw that Newton tacitly assumed the no-slip condition in the analysis of vortex motion but he missed it in the problem of the cylinder moving along its length. Daniel Bernoulli recognized as early as in 1738 that a fluid could not slip freely over a solid surface. The well known Bernoulli equation giving a relation between pressure and velocity in a fluid is valid only for an inviscid or frictionless fluid. Based on the discrepancies between the measured data for a real fluid and the calculated data for an ideal fluid he concluded that perfect slip was not possible. But it is going only half way; it does not mean no-slip was meant. Based on the observation of water flow in a channel Du Buat concluded in 1786 that the fluid adjacent to the surface was at rest, but with a qualification that this is subject to the condition that flow velocity in the channel is *sufficiently* small. This is a brave conclusion in spite of the cautious qualification. It is quite possible he was influenced by the model of a rolling ball (see section 2) which may roll without slipping at low velocities but may slip at higher velocities.

Coulomb addressed this problem also and his experiments were brilliant and were a logical extension of his experiments on dry friction. He took a metallic disk oscillating in water and smeared it with grease and later covered the grease with powdered sandstone. To his surprise the resistance of the disk scarcely changed in either case. This is a remarkable result. It appears strange initially since our intuition is based mainly on friction between solid surfaces. We know that if we grease them the friction decreases. Polishing the surfaces also leads to a decrease in friction. Coulomb might have expected that a greased surface leads to better slip. The result Coulomb arrived at is surprising but is similar to what we have seen for the Hagen-Poiseuille Flow. We saw that the resistance coefficient l = 64/Re and it depended only on Reynolds number Re but not on the surface conditions. This is not any less surprising. We may add, however, that l does depend on the surface roughness (higher l for rougher surfaces) for turbulent flows.

Notice that conclusions mentioned above and arrived at during the eighteenth century by Bernoulli, Du Buat and Coulomb came from experimental observations and before the N-S equations were known. During the nineteenth century three different hypotheses were put forward by various authors at different times. They will be discussed in the next section.

9. The Struggle to No-Slip

The first of the three hypotheses we are going to discuss assumes that velocity of the fluid at the wall is the same as that of the moving surface itself and it changes continuously inside the fluid. This is the no-slip condition and it seems to have been Coulomb's belief. The second one was put forward during the second decade of the nineteenth century by Girard who did experiments on the flow of liquids through tubes. He supposed that a very thin layer of fluid remains attached to the walls and the bulk of the fluid slips over the outer surface of this stagnant layer. Further, he supposed that if the wall material remains the same the thickness of the stagnant layer is constant. This means that this layer presents to the moving fluid the same irregularities as those of the wall itself. Because of the choice of such a model he was obliged to make other assumptions. For a liquid such as mercury that does not wet the glass tube wall, he supposed that the thickness of the layer was zero and the liquid slips over the surface. It is not too surprising how the ideas of wettability and no-slip got mixed up even though they are distinct. Wettability is related to surface tension and comes into picture only when there is a free surface. No-slip, on the other hand, is connected to viscosity and does not need a free surface. Note that the presence of a stagnant layer with slip leads to a discontinuity in velocity in the fluid. Now we know that discontinuities cannot exist in a real (or viscous) fluid since it leads to infinite stress. But this model was proposed before the N-S equations were known.

The third hypothesis was due to Navier himself. From the molecular hypothesis which led him to the correct equations of motion he deduced in 1823 that there is (partial) slipping at a solid boundary. He argued that the wall resists this slipping with a force proportional to the slip velocity. Since this tangential stress has to be continuous from the solid wall to the fluid he assumed for flow in one direction

$$bu = m\partial u / \partial n \tag{9.1}$$

where n is along the normal to the wall and b is a constant with m/b being a length. This length is zero if there is no slip. Navier explained Girard's experimental results for flow through tubes using this model. Note that there is no velocity discontinuity inside the fluid in this model.

It is interesting to note here that Poisson obtained similar conditions as Navier's but suggested that these should be applied at the outside of a stagnant layer. Stokes, another giant and who independently derived the equations of motion, was initially inclined towards the first (i.e. the correct no-slip) hypothesis but then wavered between this hypothesis and Navier's. It was because his calculations did not agree with the experimental data for pipe flow known to him. His calculations were correct and they would have agreed with the experimental results of either Hagen's or Posieuille's. In his report to the British Association in 1846 he mentioned all three hypotheses without picking any one. But finally he decided on the first (i.e. no-slip) based on two arguments – (i) Existence of slip would imply that the friction between a solid and fluid was of a different nature from, and infinitely less than, the friction between two layers of fluid. (ii) Satisfactory agreement between the results obtained with no-slip assumption and the observations.

The first argument here is remarkable. A tangential stress inside a fluid leads to a deformation of the fluid element but still the velocity is continuous (as is known from the observations). Why should it not be true at the solid-fluid interface also in the presence of friction between a solid and the fluid? For a given stress a larger deformation results if the viscosity is small. If we get a discontinuity in velocity at the interface that means the mechanism of friction between the solid and the fluid should be different and also it should be infinitely less than between two layers of fluids. Looking back it was a convincing argument from Stokes. But the no-slip condition seems to have appeared unnatural and the competing ideas had their own supporters. We have seen how Hagen and Poiseuille did the experiments but did not zero in on the no-slip condition. Even Darcy (1858) and Helmholtz (1860) settled for some form of slip!

By the end of the nineteenth century the hypothesis supported by Stokes was accepted. Finally it had to be, of course, because it is true. But to achieve that there were discussions at length. Many experiments were done and repeated. This was because we have to know if there is a small slip at the wall or is it exactly zero. Experiments on oscillating glass disks in air by Maxwell and several other experiments including flow of mercury in glass tubes were specially designed to investigate the velocity slip. Most of these experiments were concerned with the laminar flow. Noteworthy is the conclusion by Couette in 1890 that even the turbulent flows have to satisfy the no-slip condition, despite a very large gradient near the wall!

The details of the experiments by Whetham will be given in the next section which in a way settled the issue for no-slip.

10. Careful Experiments by Whetham

Whetham (1890) did a series of careful experiments to compare the time taken for a given volume of water to discharge through a glass capillary tube. After a set of measurements the capillary tube was removed, its inside silvered to form a thin smooth layer and experiments repeated. Then the silver layer was dissolved off with nitric acid. From the weight of the tube with and without silver coating, the thickness of silver deposited (about 0.014 mm) was estimated. Using Poiseuille solution for the fully developed flow correction was applied for the decrease in tube diameter due to the sliver coating and more importantly for change in viscosity due to temperature variation that occurred between two experiments. The change in flow rate due to silver coating, after these corrections were applied, was negligibly small enough to declare that there is no slip at the wall.

It is to the credit of Whetham that he repeated the experiments of Girard (1813-1815) who had measured flow of water through copper tubes and those due to Helmholtz and Piotrowski (1860) where measurements of time periods were made for a pendulum formed by a glass bulb filled with different liquids and suspended bifilarly by a fine copper wire. Due to friction at the inside wall of the bulb the Pendulum motion slowed down and its logarithmic decrement was evaluated. The experiment was repeated after the inside of the bulb was silver coated. By carefully repeating these two experiments Whetham removed the doubts that these experiments had supported a slip.

It is interesting to note that during this period when still some doubts existed on the basic issue of fluid slip at the wall, a fundamental experiment was done by Osborne Reynolds (1883) to determine when a laminar flow in a pipe became turbulent. Whetham was aware of these results and took care to keep the flow laminar in the tubes he used.

11. Navier, Maxwell and the Molecular Theory

Till now we have assumed that the continuum theory is valid and hence the molecular structure of the fluid is ignored. Then the fluid slip at the solid wall is zero. But what really happens at the molecular level?

The characteristic dimension in the dynamics of the molecules is the mean free path Λ. It is the average distance travelled by a molecule between two molecular collisions. Recall that Navier (1823), through a molecular hypothesis had concluded that slipping takes place at the wall and the length scale involved in which it takes place is m/b. Maxwell (1879) who has done pioneering work in the kinetic theory of gases, concluded that slip takes place according to the equation of Navier and the length m/b is comparable to Λ, and it may be 2Λ. In the continuum theory Λ is zero. But in a real gas Λ is non-zero but extremely small. In air at normal atmospheric temperature and pressure Λ is $0.065\,mm$. In liquids it is still smaller. This non-zero but small value of Λ was where probably one faced the difficulty, both conceptual and practical.

If Λ turns out to be comparable to the characteristic flow dimension L, say pipe radius, then slip at the wall cannot be neglected and also it is easily detected. The ratio Λ/L is the Knudsen number K_n. It is possible to increase Λ (and K_n increases as a result) by decreasing the density of the gas. For $K_n < 0.01$ one gets the continuum flow and for $K_n > 0.01$ the molecular scales do become important, continuum theory breaks down and slip cannot be ignored. These features are highlighted in figure 5. These results for rarefied gas flow are due to S.Tison (1995) and from the book by Karniadakis & Beskok (2002). Here the mass flow rate in a 2 mm diameter tube of 200 mm length is plotted for inlet and outlet pressure variation as shown. Both P_{in} and P_{out} are varied in this experiment and hence it is not possible to replot this graph in the standard form of figure 4. But what is interesting here is that the Knudsen number range is shown and we can clearly identify a shift in the type of flow and also the presence of slip when it occurs. On the right side for $K_n < 0.6$ we have the continuum flow and as we move left and if the pressure square difference falls below 500 Pa^2, the decrease in mass flow rate is less rapid. This is because of the slip at the wall due to a large mean free path Λ or a large value of K_n. In the free molecular flow regime for $K_n > 17$ the variation in mass flow rate is again linear but with a reduced slope. It is instructive to imagine these flows with large Λ. This figure covers the entire laminar flow regime. High K_n flows with slip are possible in modern engineering applications like MEMS (Micro Electro-Mechanical Systems).

12. No-Slip in Turbulent Flows

It was mentioned earlier that Couette concluded that the no-slip condition is valid for the turbulent flows also. Turbulent flows are essentially unsteady and also have a large velocity gradient near the wall. But at every instant they have to satisfy the same wall boundary conditions that a laminar flow does. Turbulent quantities are often decomposed into a time averaged value and a fluctuation; e.g. the instantaneous velocity component $u(x, y, t)$ parallel to the wall can be written as a sum of the mean and the fluctuation –

$$u(x, y, t) = \tilde{u}(x, y) + u'(x, y, t). \tag{12.1}$$

If $u(x, y, t)$ satisfies the no-slip condition, it is easy to see that so should $\tilde{u}$ and u'. Similar arguments apply to the normal velocity component $v(x, y, t)$. Hence turbulent velocity fluctuations, no matter how severe, get suppressed at the wall. If the flow near the wall is unidirectional and nearly parallel to the wall, (i.e. boundary layer-type flow) remarkable similarity rules exist for the mean velocity distribution in this flow. Such similarity in the wall layers is very unlikely if there were slip at the wall.

We saw in section 8 that friction at a wall, for example resistance coefficient l for pipes, does not depend on the wall conditions for laminar flows and also for turbulent flows if the surface is sufficiently smooth. But rough surfaces in turbulent flows lead to a larger friction.

13. Flow over a Permeable Surface and Associated Slip Velocity

Imagine the channel flow we considered in section 5 to be consisting of permeable or porous channel walls as shown in figure 6. The applied pressure gradient dp/dx induces a flow in the channel and also in the channel walls along x-direction. Even though the no-slip condition is valid on the walls of the individual pores, a slip velocity occurs in the average sense at the interface of the channel wall due to the tangential velocity in the wall. Hence it is convenient to approximate a slip boundary condition rather than consider the flow details inside the porous paths. Interesting experiments have been done by Beavers & Joseph (1967) to model such a flow.

In flow inside a permeable material like sand the filter velocity u_f is governed by the Darcy's law

$$u_f = -\frac{k}{m}\frac{dp}{dx} \tag{13.1}$$

where k is the permeability of the porous material. The true velocity of the fluid satisfies no-slip condition at the walls of the porous paths and also in some locations is bound to be higher than u_f.

Now returning back to the channel flow with a permeable wall, the flow inside the channel can be assumed to have a slip velocity u_s. Inside the channel wall as one moves away from the interface the velocity decreases from u_s to Darcy value u_f given by the equation above. With this picture in mind we would like to solve for the flow in the channel that has permeable walls. The slip velocity u_s is an unknown and it is modelled to be related to the flow inside the channel by

$$\left.\frac{du}{dy}\right|_{y=0} = \frac{a}{\sqrt{k}}(u_s - u_f) \tag{13.2}$$

where a is a dimensionless constant depending on the material parameters of the permeable wall. The solution leads to a flow rate higher than the usual fully developed case. In other words the resistance co-efficient $l = 64/\mathrm{Re}$ we came across in section 5 will be modified as indicated by equation (5.9). Here $\Delta u_w (= u_s)$ depends on a and k as described in Beavers & Joseph (1967).

It is interesting to compare the expressions for the slip velocity given by equation (9.1) due to Navier and equation (13.2) for the permeable wall with $u_f = 0$. Both the expressions model slip velocity to be proportional to the local velocity gradient or the shear stress.

14. Confidence in No-Slip

We have seen that in the continuum theory the slip at the wall is exactly zero. From the molecular theory it is known that the slip is extremely small and takes place in a length scale of the order of mean free path Λ. But it cannot be verified by direct observations and the experimental justifications have been indirect and also partly depend on some kind of modelling, e.g. N-S equations. Additionally the experiments have their own error bands. Hence in the true scientific spirit the question – How much confidence should we place in the no-slip boundary condition? – cannot be brushed aside.

If we accept this question to be pertinent the immediate question that follows is about the validity of the N-S equations themselves. A major assumption in the derivation of the N-S equations is the relation (linear, for Newtonian fluids) between the stress and rate of strain. Our experimental verification of this relation is, at best, in the same class as the experimental “proofs” we have discussed in this article about the no-slip boundary condition. Hence the individual experiments cannot help us to justify the N-S equations. Truesdell (1974) gives the similitude arguments to justify the N-S equations and we can extend them in the present context.

It is proper here to consider the N-S equations clubbed with the no-slip boundary conditions to be the model under scrutiny. In many examples the boundary conditions at the wall play a dominant role. Our confidence in this combined system comes from the rules of similitude this system satisfies. Criteria in terms of non-dimensional parameters like the Reynolds number, Mach number etc. have proved themselves valid in a variety of circumstances including in turbulent flows. If there were a partial slip according to Navier’s hypothesis another length scale m/b enters the equations in addition to the length d specifying the system. This length m/b would have been detected in the similitude tests unless m/b is zero or so small that its effects are negligible as we have seen. It is these collective experimental observations that should give us great confidence in the validity of (the N-S equations and) the no-slip condition at the wall.

Acknowledgement : Part of the work in this study was made when one of the authors (PS from IIT, Kharagpur) was supported as a Summer Research Fellow at NAL by the Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore during May -July 2001. We would like to thank Dr. R. Kidambi for helpful comments and Mr. B.G. Srinidhi, Mrs. D. Shobha, Mrs. L. Vijayalakshmi and Mr. Mallikarjun for their help in the preparation of the manuscript.

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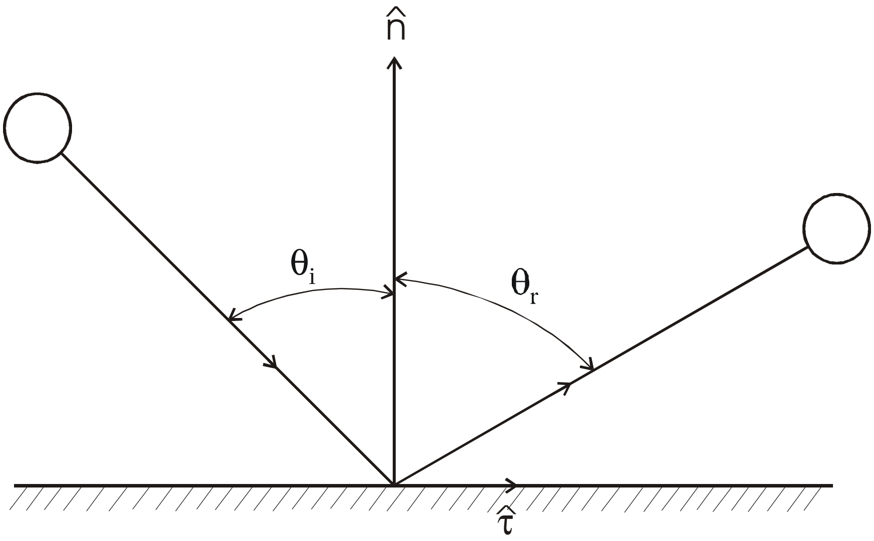

Figure 1. A ball impinging on a wall

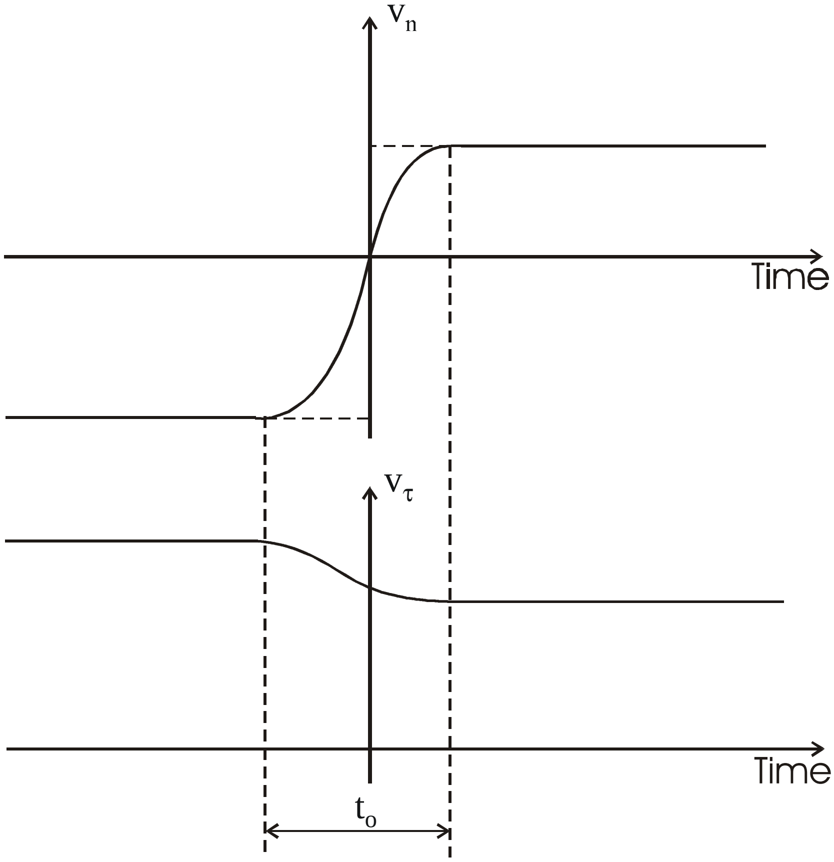

Figure 2. Time variation of Normal and Tangential velocity components of the impinging ball.

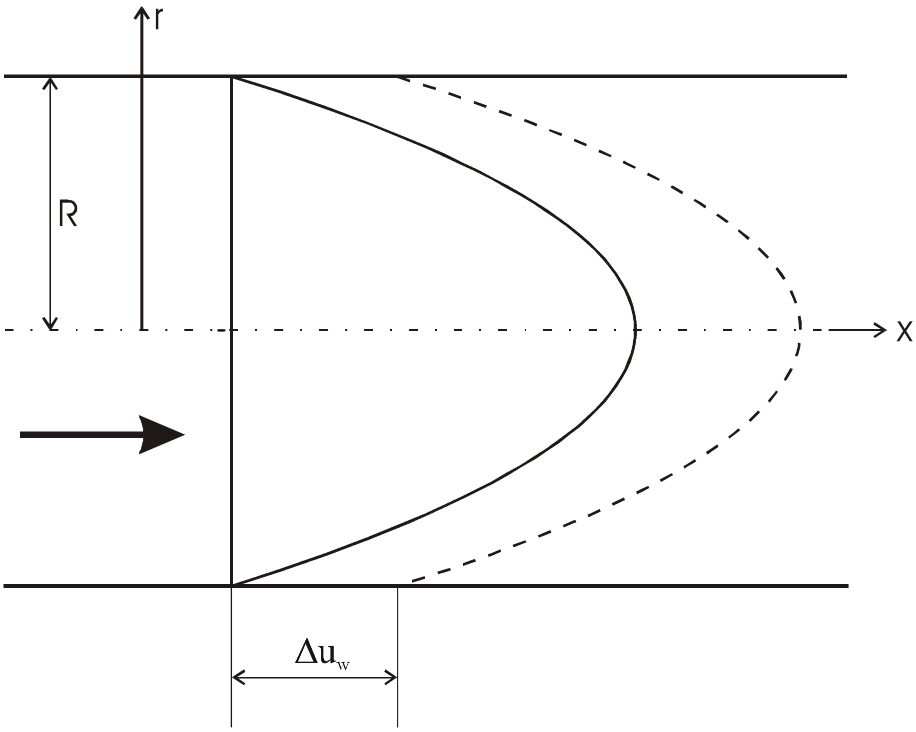

Figure 3. Parabolic velocity profile in a fully developed pipe flow with and without slip.

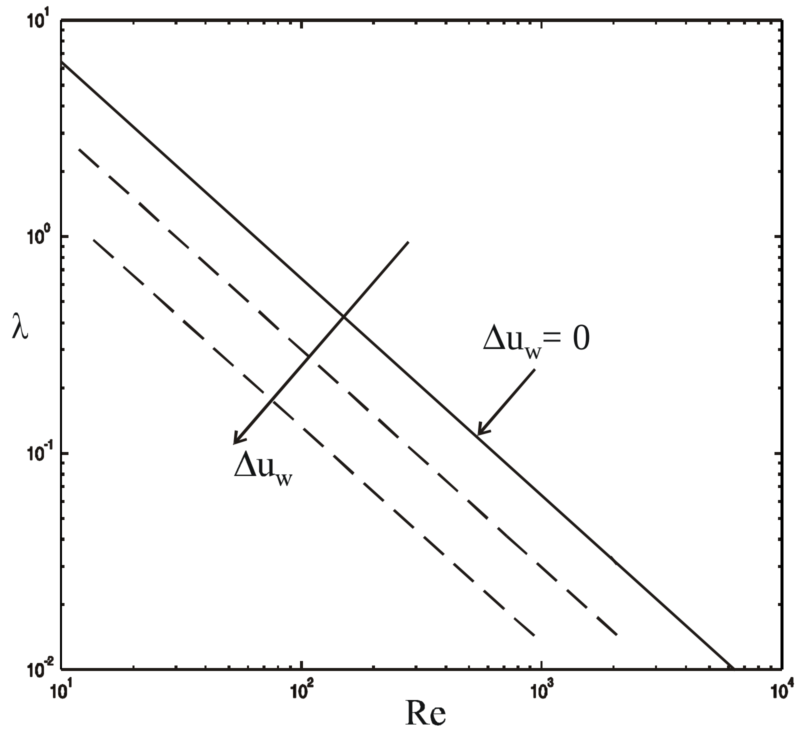

Figure 4. Variation of resistance coefficient λ as a function of Re for a fully developed pipe flow with and without slip.

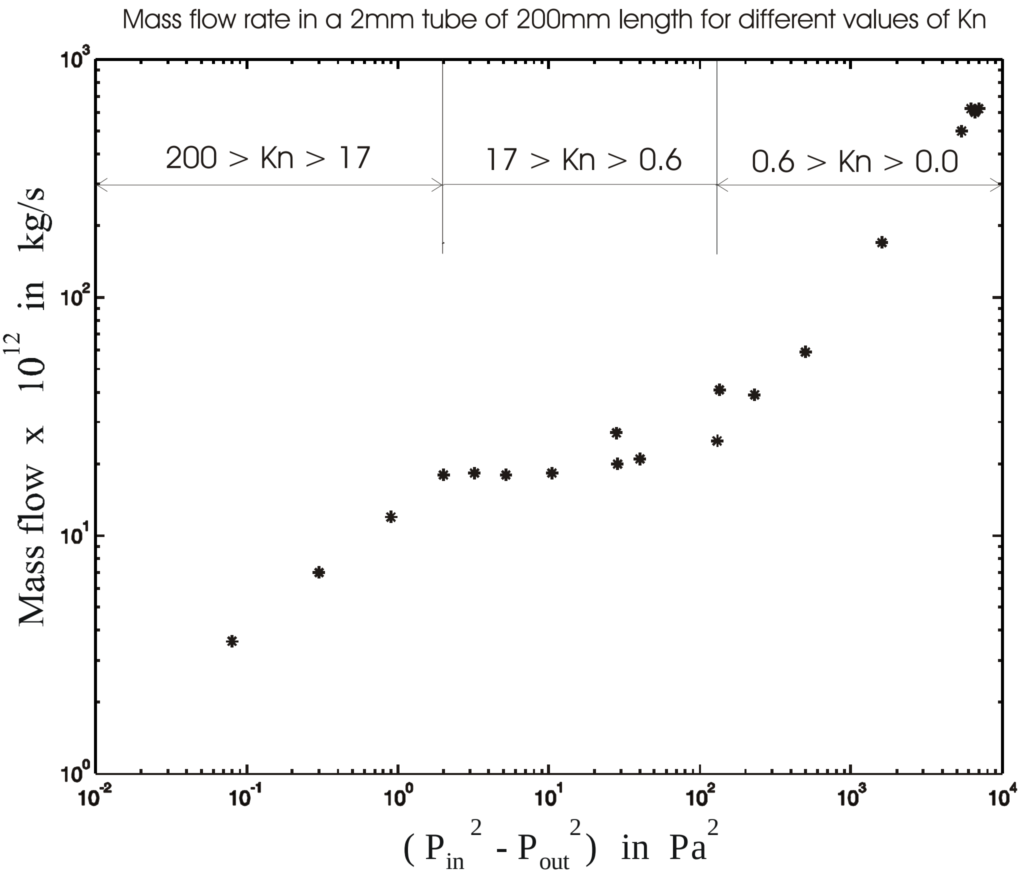

Figure 5. Mass flow rate for a rarefied gas flow in a tube as a function of ($P_{in}^2 - P_{out}^2$).Data obtained by S.Tison at NIST(Kn is based on P_{out}).

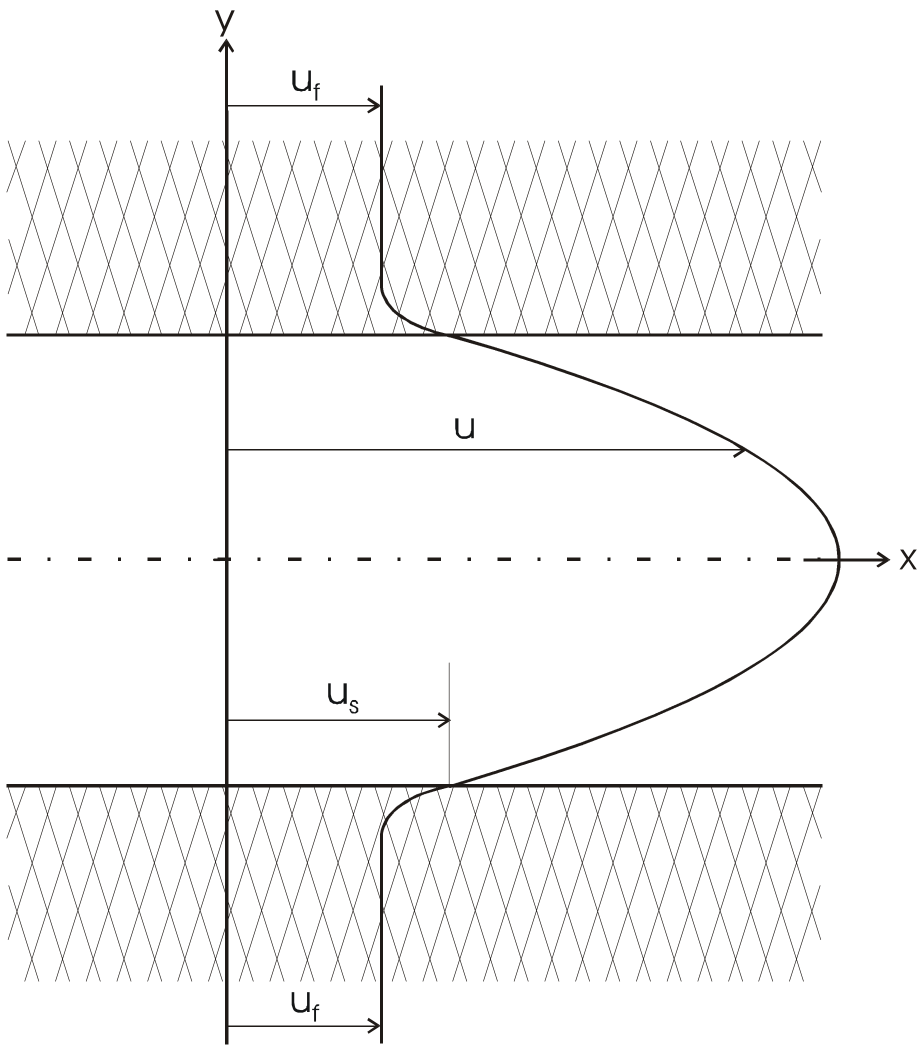

Figure 6. Flow in a channel with permeable walls. u_f is the filter velocity in the permeable material and u_s is the equivalent slip velocity.